

UNIFORM BOUNDEDNESS OF PRETANGENT SPACES AND LOCAL STRONG ONE-SIDE POROSITY

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Abstract. Let (X, d, p) be a pointed metric space. A pretangent space to X at p is a metric space consisting of some equivalence classes of convergent to p sequences $(x_n), x_n \in X$, whose degree of convergence is comparable with a given scaling sequence $(r_n), r_n \downarrow 0$. We say that (r_n) is normal if there is (x_n) such that $|d(x_n, p) - r_n| = o(r_n)$ for $n \rightarrow \infty$. Let $\Omega_p^X(\mathbf{n})$ be the set of pretangent spaces to X at p with normal scaling sequences. We prove that the spaces from $\Omega_p^X(\mathbf{n})$ are uniformly bounded if and only if $\{d(x, p) : x \in X\}$ is a so-called completely strongly porous set.

Key words: metric space, tangent space to metric space, boundedness, local strong one-side porosity.

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1 Introduction

The pretangent and tangent spaces to a metric space (X, d) at a given point $p \in X$ were defined in [5] (see also [6]). For convenience we recall some results and terminology related to the pretangent spaces.

Let (X, d, p) be a pointed metric space with a metric d and a marked point p . Fix a sequence \tilde{r} of positive real numbers r_n tending to zero. In what follows \tilde{r} will be called a *scaling sequence*. Let us denote by \tilde{X} the set of all sequences of points from X .

Definition 1.1. Two sequences $\tilde{x} = (x_n)_{n \in \mathbb{N}}$ and $\tilde{y} = (y_n)_{n \in \mathbb{N}}$, $\tilde{x}, \tilde{y} \in \tilde{X}$ are *mutually stable with respect to $\tilde{r} = (r_n)_{n \in \mathbb{N}}$* if the finite limit

$$\lim_{n \rightarrow \infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}) \quad (1.1)$$

exists.

We shall say that a family $\tilde{F} \subseteq \tilde{X}$ is *self-stable* (w.r.t. \tilde{r}) if any two $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable. A family $\tilde{F} \subseteq \tilde{X}$ is *maximal self-stable* if \tilde{F} is self-stable and for an arbitrary $\tilde{z} \in \tilde{X}$ either $\tilde{z} \in \tilde{F}$ or there is $\tilde{x} \in \tilde{F}$ such that \tilde{x} and \tilde{z} are not mutually stable.

A standard application of Zorn's lemma leads to the following

Proposition 1.2. *Let (X, d, p) be a pointed metric space. Then for every scaling sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ there exists a maximal self-stable family $\tilde{X}_{p, \tilde{r}}$ such that $\tilde{p} := (p, p, \dots) \in \tilde{X}_{p, \tilde{r}}$.*

Note that the condition $\tilde{p} \in \tilde{X}_{p, \tilde{r}}$ implies the equality $\lim_{n \rightarrow \infty} d(x_n, p) = 0$ for every $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{X}_{p, \tilde{r}}$.

Consider a function $\tilde{d} : \tilde{X}_{p, \tilde{r}} \times \tilde{X}_{p, \tilde{r}} \rightarrow \mathbb{R}$ where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y})$ is defined by (1.1). Obviously, \tilde{d} is symmetric and nonnegative. Moreover, the triangle inequality for d implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}_{p, \tilde{r}}$. Hence $(\tilde{X}_{p, \tilde{r}}, \tilde{d})$ is a pseudometric space.

Definition 1.3. *A pretangent space to the space X (at the point p w.r.t. \tilde{r}) is the metric identification of a pseudometric space $(\tilde{X}_{p, \tilde{r}}, \tilde{d})$.*

Since the notion of pretangent space is basic for the paper, we remind this metric identification construction.

Define a relation \sim on $\tilde{X}_{p, \tilde{r}}$ as $\tilde{x} \sim \tilde{y}$ if and only if $\tilde{d}(\tilde{x}, \tilde{y}) = 0$. Then \sim is an equivalence relation. Let us denote by $\Omega_{p, \tilde{r}}^X$ the set of equivalence classes in $\tilde{X}_{p, \tilde{r}}$ under the equivalence relation \sim . It follows from general properties of the pseudometric spaces (see, for example, [8]) that if ρ is defined on $\Omega_{p, \tilde{r}}^X$ as

$$\rho(\gamma, \beta) := \tilde{d}(\tilde{x}, \tilde{y}) \quad (1.2)$$

for $\tilde{x} \in \gamma$ and $\tilde{y} \in \beta$, then ρ is a well-defined metric on $\Omega_{p, \tilde{r}}^X$. By definition, the metric identification of $(\tilde{X}_{p, \tilde{r}}, \tilde{d})$ is the metric space $(\Omega_{p, \tilde{r}}^X, \rho)$.

It should be observed that $\Omega_{p, \tilde{r}}^X \neq \emptyset$ because the constant sequence \tilde{p} belongs to $\tilde{X}_{p, \tilde{r}}$. Thus every pretangent space $\Omega_{p, \tilde{r}}^X$ is a pointed metric space with the natural distinguished point $\alpha = \pi(\tilde{p})$, (see diagram (1.3) below).

Let $(n_k)_{k \in \mathbb{N}}$ be an infinite, strictly increasing sequence of natural numbers. Let us denote by \tilde{r}' the subsequence $(r_{n_k})_{k \in \mathbb{N}}$ of the scaling sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ and let $\tilde{x}' := (x_{n_k})_{k \in \mathbb{N}}$ for every $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{X}$. It is clear that if \tilde{x} and \tilde{y} are mutually stable w.r.t. \tilde{r} , then \tilde{x}' and \tilde{y}' are mutually stable w.r.t. \tilde{r}' and $\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}')$. If $\tilde{X}_{p, \tilde{r}}$ is a maximal self-stable (w.r.t \tilde{r}) family, then, by Zorn's Lemma, there exists a maximal self-stable (w.r.t \tilde{r}') family $\tilde{X}_{p, \tilde{r}'}$ such that

$$\{\tilde{x}' : \tilde{x} \in \tilde{X}_{p, \tilde{r}}\} \subseteq \tilde{X}_{p, \tilde{r}'}.$$

Denote by $in_{\tilde{r}'}$ the map from $\tilde{X}_{p,\tilde{r}}$ to $\tilde{X}_{p,\tilde{r}'}$ with $in_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{p,\tilde{r}}$. It follows from (1.2) that after metric identifications $in_{\tilde{r}'}$ passes to an isometric embedding $em' : \Omega_{p,\tilde{r}}^X \rightarrow \Omega_{p,\tilde{r}'}^X$ under which the diagram

$$\begin{array}{ccc} \tilde{X}_{p,\tilde{r}} & \xrightarrow{in_{\tilde{r}'}} & \tilde{X}_{p,\tilde{r}'} \\ \pi \downarrow & & \downarrow \pi' \\ \Omega_{p,\tilde{r}}^X & \xrightarrow{em'} & \Omega_{p,\tilde{r}'}^X \end{array} \quad (1.3)$$

is commutative. Here π and π' are the natural projections,

$$\pi(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{p,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\} \quad \text{and} \quad \pi'(\tilde{x}') := \{\tilde{y}' \in \tilde{X}_{p,\tilde{r}'} : \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}') = 0\}.$$

Let X and Y be metric spaces. Recall that a map $f : X \rightarrow Y$ is called an *isometry* if f is distance-preserving and onto.

Definition 1.4. A pretangent $\Omega_{p,\tilde{r}}^X$ is tangent if $em' : \Omega_{p,\tilde{r}}^X \rightarrow \Omega_{p,\tilde{r}'}^X$ is an isometry for every \tilde{r}' .

The following lemmas will be used in sections 3 and 4 of the paper.

Lemma 1.5. [1] Let \mathfrak{B} be a countable subfamily of \tilde{X} and let $\tilde{\rho} = (\rho_n)_{n \in \mathbb{N}}$ be a scaling sequence. Suppose that the inequality

$$\limsup_{n \rightarrow \infty} \frac{d(b_n, p)}{\rho_n} < \infty$$

holds for every $\tilde{b} = (b_n)_{n \in \mathbb{N}} \in \mathfrak{B}$. Then there is an infinite subsequence $\tilde{\rho}'$ of $\tilde{\rho}$ such that the family $\mathfrak{B}' = \{\tilde{b}' : \tilde{b} \in \mathfrak{B}'\}$ is self-stable w.r.t. $\tilde{\rho}'$.

The next lemma follows from Corollary 3.3 of [7].

Lemma 1.6. Let (X, d) be a metric space and let Y, Z be dense subsets of X . Then for every $p \in Y \cap Z$ and every $\Omega_{p,\tilde{r}}^Y$ there are $\Omega_{p,\tilde{r}}^Z$ and an isometry $f : \Omega_{p,\tilde{r}}^Y \rightarrow \Omega_{p,\tilde{r}}^Z$ such that $f(\alpha_Y) = \alpha_Z$ where α_Y and α_Z are the marked points of $\Omega_{p,\tilde{r}}^Y$ and $\Omega_{p,\tilde{r}}^Z$ respectively.

It was proved in [1] that a bounded tangent space to X at p exists if and only if the distance set

$$S_p(X) = \{d(x, p) : x \in X\}$$

is strongly porous at 0. The necessary and sufficient conditions under which all pretangent spaces to X at p are bounded also formulated in terms of the local porosity of the set $S_p(X)$ (see [2] for details).

In the present paper we shall consider some interconnections between pretangent spaces and a subclass of the locally strongly porous on the right sets.

2 Completely strongly porous sets

Let us recall the definition of the right hand porosity. Let E be a subset of $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. [10] *The right hand porosity of E at 0 is the quantity*

$$p^+(E, 0) := \limsup_{h \rightarrow 0^+} \frac{\lambda(E, 0, h)}{h} \quad (2.1)$$

where $\lambda(E, 0, h)$ is the length of the largest open subinterval of $(0, h)$ that contains no points of E . The set E is strongly porous at 0 if $p^+(E, 0) = 1$.

Let $\tilde{\tau} = (\tau_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. We shall say that $\tilde{\tau}$ is almost decreasing if the inequality $\tau_{n+1} \leq \tau_n$ holds for sufficiently large n . Write \tilde{E}_0^d for the set of almost decreasing sequences $\tilde{\tau}$ with $\lim_{n \rightarrow \infty} \tau_n = 0$ and $\tau_n \in E \setminus \{0\}$ for $n \in \mathbb{N}$.

Define \tilde{I}_E^d to be the set of sequences of open intervals $(a_n, b_n) \subseteq \mathbb{R}^+, n \in \mathbb{N}$, meeting the conditions:

- Each (a_n, b_n) is a connected component of the set $\text{Ext}E = \text{Int}(\mathbb{R}^+ \setminus E)$, i.e., $(a_n, b_n) \cap E = \emptyset$ but

$$((a, b) \neq (a_n, b_n)) \Rightarrow ((a, b) \cap E \neq \emptyset)$$

for every $(a, b) \supseteq (a_n, b_n)$;

- $(a_n)_{n \in \mathbb{N}}$ is almost decreasing;
- $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} \frac{b_n - a_n}{b_n} = 1$.

Define also an equivalence \asymp on the set of sequences of strictly positive numbers as follows. Let $\tilde{a} = (a_n)_{n \in \mathbb{N}}$ and $\tilde{\gamma} = (\gamma_n)_{n \in \mathbb{N}}$. Then $\tilde{a} \asymp \tilde{\gamma}$ if there are some constants $c_1, c_2 > 0$ such that $c_1 a_n < \gamma_n < c_2 a_n$ for $n \in \mathbb{N}$.

Definition 2.2. Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} \in \tilde{E}_0^d$. The set E is $\tilde{\tau}$ -strongly porous at 0 if there is a sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ such that $\tilde{\gamma} \asymp \tilde{a}$ where $\tilde{a} = (a_n)_{n \in \mathbb{N}}$. E is completely strongly porous at 0 if E is $\tilde{\tau}$ -strongly porous at 0 for every $\tilde{\tau} \in \tilde{E}_0^d$.

The last definition is an equivalent form of Definition 2.4 from [4] (that follows directly from Lemma 2.11 in [4]). We denote by **CSP** the set of all completely strongly porous at 0 subsets of \mathbb{R}^+ . It is clear that every $E \in \mathbf{CSP}$ is strongly porous at 0 but not conversely. Moreover, if 0 is an isolated point of $E \subseteq \mathbb{R}^+$, then $E \in \mathbf{CSP}$.

The next lemma immediately follows from Definition 2.2.

Lemma 2.3. *Let $E \subseteq \mathbb{R}^+$, $\tilde{\gamma} \in \tilde{E}_0^d$, $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ and let $\tilde{a} = (a_n)_{n \in \mathbb{N}}$. The equivalence $\tilde{\gamma} \asymp \tilde{a}$ holds if and only if we have*

$$\limsup_{n \rightarrow \infty} \frac{a_n}{\gamma_n} < \infty \quad \text{and} \quad \gamma_n \leq a_n$$

for sufficiently large n .

Define a set \mathbb{N}_{N_1} as $\{N_1, N_1 + 1, \dots\}$ for $N_1 \in \mathbb{N}$.

Definition 2.4. *Let*

$$\tilde{A} = \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d \quad \text{and} \quad \tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d.$$

Write $\tilde{A} \preceq \tilde{L}$ if there are $N_1 \in \mathbb{N}$ and $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$ such that $a_n = l_{f(n)}$ for every $n \in \mathbb{N}_{N_1}$. \tilde{L} is universal if $\tilde{B} \preceq \tilde{L}$ holds for every $\tilde{B} \in \tilde{I}_E^d$.

Let $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ be universal and let

$$M(\tilde{L}) := \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}}. \quad (2.2)$$

In what follows acE means the set of all accumulation points of a set E .

Theorem 2.5. [4] *Let $E \subseteq \mathbb{R}^+$ be strongly porous at 0 and $0 \in acE$. Then $E \in \mathbf{CSP}$ if and only if there is an universal $\tilde{L} \in \tilde{I}_E^d$ such that $M(\tilde{L}) < \infty$.*

Note that the quantity $M(\tilde{L})$ depends from the set E only (for details see [4]). The following lemma is used in next part of the paper.

Lemma 2.6. [4] *Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} = (\tau_n)_{n \in \mathbb{N}} \in \tilde{E}_0^d$. Then E is $\tilde{\tau}$ -strongly porous at 0 if and only if there is a constant $k \in (1, \infty)$ such that for every $K \in (k, \infty)$ there exists $N_1(K) \in \mathbb{N}$ with $(k\tau_n, K\tau_n) \cap E = \emptyset$ for every $n \geq N_1(K)$.*

Let us consider now a simple set belonging to **CSP**.

Example 2.7. Let $(x_n)_{n \in \mathbb{N}}$ be strictly decreasing sequence of positive real numbers with $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 0$. Define a set W as

$$W = \{x_n : n \in \mathbb{N}\},$$

i.e., W is the range of the sequence $(x_n)_{n \in \mathbb{N}}$. Then $W \in \mathbf{CSP}$ and $\tilde{L} = \{(x_{n+1}, x_n)\}_{n \in \mathbb{N}} \in \tilde{I}_W^d$ is universal with $M(\tilde{L}) = 1$.

Proposition 2.8. Let $E \subseteq \mathbb{R}^+$. Then the inclusion

$$\{E \cup A : A \in \mathbf{CSP}\} \subseteq \mathbf{CSP} \quad (2.3)$$

holds if and only if 0 is an isolated point of E .

Remark 2.9. Inclusion (2.3) means that $E \cup A \in \mathbf{CSP}$ for every $A \in \mathbf{CSP}$.

Proof of Proposition 2.8. If $0 \notin \text{ac}E$, then (2.3) follows almost directly and we omit the details here. Suppose $0 \in \text{ac}E$. Then there is a sequence $(\tau_n)_{n \in \mathbb{N}}$ such that $\tau_n \in E$ and $\tau_{n+1} \leq 2^{-n^2} \tau_n$ for every $n \in \mathbb{N}$. Let $M_1, M_2, \dots, M_k, \dots$ be an infinite partition of \mathbb{N} ,

$$\bigcup_{k=1}^{\infty} M_k = \mathbb{N}, \quad M_i \cap M_j = \emptyset \text{ if } i \neq j$$

such that $\text{card} M_k = \text{card} \mathbb{N}$ for every k and $\nu(1) < \nu(2) < \dots < \nu(k) \dots$ where

$$\nu(k) = \min_{n \in M_k} n. \quad (2.4)$$

Let $n \in \mathbb{N}$ and let $m(n)$ be the index such that $n \in M_{m(n)}$. For every $n \in \mathbb{N}$ define τ_n^* as $2^{-m(n)} \tau_n$. Write

$$E_1 = \{\tau_n : n \in \mathbb{N}\} \quad \text{and} \quad E_1^* = \{\tau_n^* : n \in \mathbb{N}\}.$$

Using Lemma 2.6 we can show that $E_1 \cup E_1^*$ is not τ^* -strongly porous with $\tilde{\tau}^* = (\tau_n)_{n \in \mathbb{N}}$. Consequently $E_1 \cup E_1^* \notin \mathbf{CSP}$. It implies that $E \cup E_1^* \notin \mathbf{CSP}$ because $E_1 \subseteq E$. To complete the proof, it suffices to show that $E_1^* \in \mathbf{CSP}$. To this end, we note that (2.4) and the inequalities $\nu(1) < \nu(2) < \dots < \nu(k) \dots$ imply that $m(n) \leq n$ for every $n \in \mathbb{N}$. Indeed, if $m(n) = k$, then we have

$$\begin{aligned} n &\geq \nu(k) = (\nu(k) - \nu(k-1)) + (\nu(k-1) - \nu(k-2)) + \dots + (\nu(2) - \nu(1)) + \nu(1) \\ &\geq (k-1) + \nu(1) = k = m(n). \end{aligned}$$

Consequently

$$\tau_n^* = 2^{-m(n)} \tau_n \geq 2^{-n} \tau_n \geq 2^{-n^2} \tau_n \geq \tau_{n+1} \geq \tau_{n+1}^*$$

for every $n \in \mathbb{N}$. It follows from that

$$\lim_{n \rightarrow \infty} \frac{\tau_n^*}{\tau_{n+1}^*} \geq \lim_{n \rightarrow \infty} \frac{2^{-n} \tau_n}{2^{-n^2} \tau_n} = \lim_{n \rightarrow \infty} 2^{n^2 - n} = +\infty.$$

Thus, as in Example 2.7, we have $E_1^* \in \mathbf{CSP}$. \square

3 Uniform boundedness and \mathbf{CSP}

Let $\mathfrak{F} = \{(X_i, d_i) : i \in I\}$ be a nonempty family of metric spaces. The family \mathfrak{F} is *uniformly bounded* if there is a constant $c > 0$ such that the inequality $\text{diam} X_i < c$ holds for every $i \in I$. If all metric spaces $(X_i, d_i) \in \mathfrak{F}$ are pointed with marked points $p_i \in X_i$, then the uniform boundedness of \mathfrak{F} can be described by the next way. Define

$$\rho^*(X_i) := \sup_{x \in X_i} d_i(x, p_i) \quad \text{and} \quad R^*(\mathfrak{F}) := \sup_{i \in I} \rho^*(X_i). \quad (3.1)$$

The family \mathfrak{F} is uniformly bounded if and only if $R^*(\mathfrak{F}) < \infty$.

Proposition 3.1. *Let (X, d, p) be a pointed metric space and let $\Omega_{\mathbf{p}}^{\mathbf{X}}$ be the set of all pretangent spaces to X at p . The following statements are equivalent.*

- (i₁) *The family $\Omega_{\mathbf{p}}^{\mathbf{X}}$ is uniformly bounded.*
- (i₂) *The point p is an isolated point of X .*

Proof. The implication (i₂) \Rightarrow (i₁) follows directly from the definitions. To prove (i₁) \Rightarrow (i₂) suppose that $p \in \text{ac} X$. Let $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{X}$ be a sequence of distinct points of X such that $\lim_{n \rightarrow \infty} d(x_n, p) = 0$. For $t > 0$ define the scaling sequence $\tilde{r}_t = (r_{n,t})_{n \in \mathbb{N}}$ with $r_{n,t} = \frac{d(x_n, p)}{t}$. It follows at once from Definition 1.1 that \tilde{x} and \tilde{p} are mutually stable w.r.t \tilde{r}_t and

$$\tilde{d}_{\tilde{r}_t}(\tilde{x}, \tilde{p}) = t. \quad (3.2)$$

Let $\tilde{X}_{p, \tilde{r}_t}$ be a maximal self-stable family meeting the relation $\tilde{x} \in \tilde{X}_{p, \tilde{r}_t}$. Equality (3.2) implies the inequality

$$\text{diam} \Omega_{p, \tilde{r}_t}^X \geq t,$$

where $\Omega_{p, \tilde{r}_t}^X = \pi(\tilde{X}_{p, \tilde{r}_t})$. Consequently the family $\Omega_{\mathbf{p}}^{\mathbf{X}}$ is not uniformly bounded. The implication (i₁) \Rightarrow (i₂) follows. \square

The proposition above shows that the question on the uniform boundedness can be informative only for some special subfamilies of $\Omega_{\mathbf{p}}^{\mathbf{X}}$. We can narrow down the family $\Omega_{\mathbf{p}}^{\mathbf{X}}$ by the way of consideration some special scaling sequences.

Definition 3.2. Let (X, d, p) be a pointed metric space and let $p \in \text{ac}X$. A scaling sequence $(r_n)_{n \in \mathbb{N}}$ is normal if there is $(x_n)_{n \in \mathbb{N}} \in \tilde{X}$ such that the sequence $(d(x_n, p))_{n \in \mathbb{N}}$ is almost decreasing and

$$\lim_{n \rightarrow \infty} \frac{d(x_n, p)}{r_n} = 1. \quad (3.3)$$

Proposition 3.3. The following properties take place for every pointed metric space (X, d, p) .

- (i₁) If $\Omega_{p, \tilde{r}}^X$ contains at least two distinct points, then there are $c > 0$ and a subsequence $(r_{n_k})_{k \in \mathbb{N}}$ of \tilde{r} so that the sequence $(cr_{n_k})_{k \in \mathbb{N}}$ is normal.
- (i₂) If $(x_n)_{n \in \mathbb{N}} \in \tilde{X}$ and (3.3) holds, then there is an infinite increasing sequence $(n_k)_{k \in \mathbb{N}}$ so that $(d(x_{n_k}, p))_{k \in \mathbb{N}}$ is decreasing.
- (i₃) If \tilde{r} is a normal scaling sequence, then there is $(x_n)_{n \in \mathbb{N}} \in \tilde{X}$ such that (3.3) holds and $(d(x_n, p))_{n \in \mathbb{N}}$ is almost decreasing.

Proof. It is easily verified that (i₁) and (i₂) hold. To verify (i₃) observe that there is $(y_n)_{n \in \mathbb{N}} \in \tilde{X}$ which satisfies $d(y_n, p) > 0$ for every $n \in \mathbb{N}$ and (3.3) with $(x_n)_{n \in \mathbb{N}} = (y_n)_{n \in \mathbb{N}}$. Let $m(n) \in \mathbb{N}$ meet the conditions $m(n) \leq n$ and $d(y_{m(n)}, p) = \min_{1 \leq i \leq n} d(y_i, p)$. Since \tilde{r} is almost decreasing we have

$$\frac{d(y_{m(n)}, p)}{r_{m(n)}} \leq \frac{d(y_{m(n)}, p)}{r_n} \leq \frac{d(y_n, p)}{r_n}.$$

The conditions $\lim_{n \rightarrow \infty} y_n = p$ and $d(y_n, p) > 0$ for $n \in \mathbb{N}$ imply that $m(n) \rightarrow \infty$ as $n \rightarrow \infty$. Consequently

$$1 = \lim_{n \rightarrow \infty} \frac{d(y_{m(n)}, p)}{r_{m(n)}} = \lim_{n \rightarrow \infty} \frac{d(y_{m(n)}, p)}{r_n} = \lim_{n \rightarrow \infty} \frac{d(y_n, p)}{r_n}.$$

Thus (i₃) holds with $(x_n)_{n \in \mathbb{N}} = (y_{m(n)})_{n \in \mathbb{N}}$. □

Write $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ for the set of pretangent spaces $\Omega_{p, \tilde{r}}^X$ with normal scaling sequences. Under what conditions the family $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly bounded?

Remark 3.4. Of course, the property of scaling sequence \tilde{r} to be normal depends on the underlying space (X, d, p) . Nevertheless for every pointed

metric space (X, d, p) a scaling sequence \tilde{r} is normal for this space if and only if it is normal for the space $(S_p(X), |\cdot|, 0)$. We shall use this simple fact below in Proposition 3.5.

In the next proposition we define $\Omega_0^E(\mathbf{n})$ to be the set of all pretangent spaces to the distance set $E = S_p(X)$ at 0 w.r.t. normal scaling sequences.

Proposition 3.5. *Let (X, d, p) be a pointed metric space and let $E = S_p(X)$. Then we have*

$$R^*(\Omega_0^E(\mathbf{n})) = R^*(\Omega_p^X(\mathbf{n})) \quad (3.4)$$

where $R^*(\Omega_p^X(\mathbf{n}))$ and $R^*(\Omega_0^E(\mathbf{n}))$ are defined by (3.1) with $\mathfrak{F} = \Omega_p^X(\mathbf{n})$ and $\mathfrak{F} = \Omega_0^E(\mathbf{n})$ respectively.

Proof. If $p \notin acX$, then the set of normal scaling sequences is empty. Consequently we have $\Omega_0^E(\mathbf{n}) = \Omega_p^X(\mathbf{n}) = \emptyset$, so we suppose that $p \in acX$.

For each normal scaling sequence \tilde{r} and every $\tilde{x} \in \tilde{X}$ having the finite limit $\lim_{n \rightarrow \infty} \frac{d(x_n, p)}{r_n}$ we can find $(s_n)_{n \in \mathbb{N}} \in \tilde{E}$ such that

$$\lim_{n \rightarrow \infty} \frac{d(x_n, p)}{r_n} = \lim_{n \rightarrow \infty} \frac{s_n}{r_n}. \quad (3.5)$$

Hence the inequality

$$\rho(\alpha, \beta) \leq R^*(\Omega_0^E(\mathbf{n})) \quad (3.6)$$

holds with $\alpha = \pi(\tilde{p})$ for every $\beta \in \Omega_{p, \tilde{r}}^X$ and every $\Omega_{p, \tilde{r}}^X \in \Omega_p^X(\mathbf{n})$. Taking supremum over all $\Omega_{p, \tilde{r}}^X \in \Omega_p^X(\mathbf{n})$ and $\beta \in \Omega_{p, \tilde{r}}^X$, we get

$$R^*(\Omega_0^E(\mathbf{n})) \geq R^*(\Omega_p^X(\mathbf{n})). \quad (3.7)$$

It still remains to prove the inequality

$$R^*(\Omega_0^E(\mathbf{n})) \leq R^*(\Omega_p^X(\mathbf{n})). \quad (3.8)$$

As is easily seen, for every normal scaling \tilde{r} and every $\tilde{s} \in \tilde{E}$ with $\lim_{n \rightarrow \infty} \frac{s_n}{r_n} < \infty$, there is $\tilde{x} \in \tilde{X}$ satisfying (3.5). Now reasoning as in the proof of (3.7) we obtain (3.8). Equality (3.4) follows from (3.7) and (3.8). \square

Lemma 3.6. *Let $E \subseteq \mathbb{R}^+$ and let $0 \in acE$. If the inequality*

$$R^*(\Omega_0^E(\mathbf{n})) < \infty \quad (3.9)$$

holds, then $E \in \mathbf{CSP}$.

Proof. Suppose that (3.9) holds but there is $\tilde{\tau} = (\tau_n)_{n \in \mathbb{N}} \in \tilde{E}_0^d$ such that E is not $\tilde{\tau}$ -strongly porous at 0. Then, by Lemma 2.6, for every $k > 1$ there is $K \in (k, \infty)$ such that $(k\tau_n, K\tau_n) \cap E \neq \emptyset$ for all n belonging to an infinite set $A \subseteq \mathbb{N}$. Let us put

$$k = 2R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})). \quad (3.10)$$

It simply follows from (3.1) and Definition 3.2 that $R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) \geq 1$. Thus $k \geq 2$. Consequently we can find $K \in (k, \infty)$ and an infinite set $A = \{n_1, \dots, n_j, \dots\} \subseteq \mathbb{N}$, such that for every $n_j \in A$ there is $x_j \in E$ satisfying the double inequality

$$k < \frac{x_j}{\tau_{n_j}} < K. \quad (3.11)$$

Thus the sequence $\left(\frac{x_j}{\tau_{n_j}}\right)_{j \in \mathbb{N}}$ is bounded. Hence it contains a convergent subsequence. Passing to this subsequence we obtain

$$\lim_{j \rightarrow \infty} \frac{x_j}{\tau_{n_j}} < \infty. \quad (3.12)$$

Now (3.10) and (3.11) imply

$$\lim_{j \rightarrow \infty} \frac{x_j}{\tau_{n_j}} = \lim_{j \rightarrow \infty} \frac{|0 - x_j|}{\tau_{n_j}} \geq 2R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})). \quad (3.13)$$

The scaling sequence $\tilde{r} = (r_j)_{j \in \mathbb{N}}$ with $r_j = \tau_{n_j}$, $j \in \mathbb{N}$, is normal. The existence of finite limit (3.12) implies that $\tilde{x} = (x_j)_{j \in \mathbb{N}}$ and $\tilde{0}$ are mutually stable w.r.t \tilde{r} . Consequently there is a maximal self-stable family $\tilde{E}_{0, \tilde{r}}$ such that $\tilde{x}, \tilde{0} \in \tilde{E}_{0, \tilde{r}}$. Write $\Omega_{0, \tilde{r}}^E$ for the metric identification of $\tilde{E}_{0, \tilde{r}}$ and α for the natural projection of $\tilde{0}$. Using (3.13) and (3.1), we obtain

$$R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) \geq \sup_{\gamma \in \Omega_{0, \tilde{r}}^E} \rho(\alpha, \gamma) \geq 2R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})).$$

The last double inequality is inconsistent because $1 \leq R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) < \infty$. Thus if (3.9) holds, then E is $\tilde{\tau}$ -strongly porous at 0, as required. \square

Let $\tilde{\tau} \in \tilde{E}_0^d$. Define a subset $\tilde{I}_E^d(\tilde{\tau})$ of the set \tilde{I}_E^d by the rule:

$$\begin{aligned} (\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d(\tilde{\tau})) &\Leftrightarrow (\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d \text{ and} \\ &\tau_n \leq a_n \text{ for sufficiently large } n \in \mathbb{N}). \end{aligned}$$

Write

$$C(\tilde{\tau}) := \inf_{n \rightarrow \infty} \left(\limsup \frac{a_n}{\tau_n} \right) \quad \text{and} \quad C_E := \sup_{\tilde{\tau} \in \tilde{E}_0^d} C(\tilde{\tau}) \quad (3.14)$$

where the infimum is taken over all $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d(\tilde{\tau})$.

Using Theorem 2.5 we can prove the following

Proposition 3.7. [4] *Let $E \subseteq \mathbb{R}^+$ and $\tilde{\tau} \in \tilde{E}_0^d$. The set E is $\tilde{\tau}$ -strongly porous at 0 if and only if $C(\tilde{\tau}) < \infty$. The membership $E \in \mathbf{CSP}$ holds if and only if $C_E < \infty$.*

Remark 3.8. If $E \subseteq \mathbb{R}^+$, $i = 1, 2$, $\{(a_n^{(i)}, b_n^{(i)})\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ and $\tilde{a}^1 \asymp \tilde{a}^2$ where $\tilde{a}^i = (a_n^{(i)})_{n \in \mathbb{N}}$, then there is $n_0 \in \mathbb{N}$ such that $(a_n^{(1)}, b_n^{(1)}) = (a_n^{(2)}, b_n^{(2)})$ for every $n \geq n_0$. Consequently if E is $\tilde{\tau}$ -strongly porous and $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d(\tilde{\tau})$, then we have

$$\text{either } \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} = \infty \quad \text{or} \quad \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} = C(\tilde{\tau}) < \infty.$$

Lemma 3.9. [4] *Let $E \in \mathbf{CSP}$. If $\tilde{L} \in \tilde{I}_E^d$ is universal, then $M(\tilde{L}) = C_E$ where the quantities $M(\tilde{L})$ and C_E are defined by (2.2) and (3.14) respectively.*

Proposition 3.10. *Let $E \subseteq \mathbb{R}^+$ and let $0 \in acE$. Then the equality*

$$C_E = R^*(\Omega_0^E(\mathbf{n})) \tag{3.15}$$

holds.

Proof. Let us prove the inequality

$$C_E \geq R^*(\Omega_0^E(\mathbf{n})). \tag{3.16}$$

This is trivial if $C_E = \infty$. Suppose that $C_E < \infty$. Inequality (3.16) holds if, for every normal scaling sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ and each $\tilde{y} = (y_n)_{n \in \mathbb{N}} \in \tilde{E}_0^d$, the existence of the finite limit $\lim_{n \rightarrow \infty} \frac{y_n}{r_n}$ implies the inequality

$$\lim_{n \rightarrow \infty} \frac{y_n}{r_n} \leq C_E. \tag{3.17}$$

Since \tilde{r} is normal, Proposition 3.3 implies that there is $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{E}_0^d$ with $\lim_{n \rightarrow \infty} \frac{r_n}{x_n} = 1$. Consequently (3.17) holds if and only if

$$\lim_{n \rightarrow \infty} \frac{y_n}{x_n} \leq C_E. \tag{3.18}$$

If $\lim_{n \rightarrow \infty} \frac{y_n}{x_n} = 0$, then (3.18) is trivial. Suppose that $0 < \lim_{n \rightarrow \infty} \frac{y_n}{x_n} < \infty$. The last double inequality implies the equivalence $\tilde{x} \asymp \tilde{y}$. In accordance with

Proposition 3.7, $E \in \mathbf{CSP}$ if and only if $C_E < \infty$ holds. Hence E is \tilde{x} -strongly porous at 0. Consequently there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ such that $\tilde{x} \asymp \tilde{a}$. The relations $\tilde{x} \asymp \tilde{y}$ and $\tilde{x} \asymp \tilde{a}$ imply $\tilde{y} \asymp \tilde{a}$. Using Lemma 2.3 we can find $N_0 \in \mathbb{N}$ such that $y_n \leq a_n$ for $n \geq N_0$. Consequently we have $\frac{y_n}{x_n} \leq \frac{a_n}{x_n}$ for $n \geq N_0$, which implies

$$\lim_{n \rightarrow \infty} \frac{y_n}{x_n} \leq \limsup_{n \rightarrow \infty} \frac{a_n}{x_n} \leq C(\tilde{\tau}) \leq C_E$$

(see (3.14)). Inequality (3.16) follows.

To prove (3.15), it still remains to verify the inequality

$$C_E \leq R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})). \quad (3.19)$$

It is trivial if $R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) = \infty$. Suppose that

$$R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) < \infty. \quad (3.20)$$

Inequality (3.19) holds if

$$C(\tilde{x}) \leq R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})). \quad (3.21)$$

for every $\tilde{x} \in \tilde{E}_0^d$. Let E^1 denote the closure of the set E in \mathbb{R}^+ and let $\tilde{x} \in \tilde{E}_0^d$. It follows at once from Lemma 1.6 that $R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) = R^*(\Omega_0^{\mathbf{E}^1}(\mathbf{n}))$. Consequently (3.21) holds if $C(\tilde{x}) \leq R^*(\Omega_0^{\mathbf{E}^1}(\mathbf{n}))$. By Lemma 3.6, inequality (3.20) implies that $E \in \mathbf{CSP}$. Hence, by Lemma 2.3, there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ such that

$$\limsup_{n \rightarrow \infty} \frac{a_n}{x_n} < \infty \quad (3.22)$$

and $a_n \geq x_n$ for sufficiently large n . Inequality (3.22) implies the equality

$$C(\tilde{x}) = \limsup_{n \rightarrow \infty} \frac{a_n}{x_n}, \quad (3.23)$$

(see Remark 3.8). Let $(n_j)_{j \in \mathbb{N}}$ be an infinite increasing sequence for which

$$\lim_{j \rightarrow \infty} \frac{a_{n_j}}{x_{n_j}} = \limsup_{n \rightarrow \infty} \frac{a_n}{x_n}. \quad (3.24)$$

Define $r_j := x_{n_j}$, $\tilde{r} := (r_j)_{j \in \mathbb{N}}$ and $t_j := a_{n_j}$, $\tilde{t} := (t_j)_{j \in \mathbb{N}}$. It is clear that \tilde{r} is a normal scaling sequence. Relation (3.22) and (3.24) imply that \tilde{t} and $\tilde{0} = (0, 0, \dots, 0, \dots)$ are mutually stable w.r.t. \tilde{r} . Let $\tilde{E}_{0, \tilde{r}}^1$ be a maximal (in \tilde{E}^1) self-stable family containing \tilde{t} and $\tilde{0}$. Using (3.1), (3.23) and (3.24), we obtain

$$R^*(\Omega_0^{\mathbf{E}^1}(\mathbf{n})) \geq \sup_{\tilde{y} \in \tilde{E}_{0, \tilde{r}}^1} \tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{0}) \geq \tilde{d}_{\tilde{r}}(\tilde{t}, \tilde{0}) = C(\tilde{x}).$$

Hence (3.19) holds that completes the proof of (3.15). \square

The following theorem gives the necessary and sufficient conditions under which $\Omega_p^X(\mathbf{n})$ is uniformly bounded.

Theorem 3.11. *Let (X, d, p) be a pointed metric space and let $E = S_p(X)$. The family $\Omega_p^X(\mathbf{n})$ is uniformly bounded if and only if $E \in \mathbf{CSP}$. If $\Omega_p^X(\mathbf{n})$ is uniformly bounded and $p \in acX$, then*

$$R^*(\Omega_p^X(\mathbf{n})) = M(\tilde{L}) \quad (3.25)$$

where \tilde{L} is an universal element of (\tilde{I}_E^d, \preceq) and $M(\tilde{L})$ is defined by (2.2).

Proof. The theorem is trivial if p is an isolated point of X , so that we assume $p \in acX$. By Proposition 3.5, $\Omega_p^X(\mathbf{n})$ is uniformly bounded if and only if $\Omega_0^E(\mathbf{n})$ is uniformly bounded. Since $C_E = R^*(\Omega_0^E(\mathbf{n}))$ (see (3.15)), $\Omega_0^E(\mathbf{n})$ is uniformly bounded if and only if $C_E < \infty$. Using Proposition 3.7 we obtain that $\Omega_0^E(\mathbf{n})$ is uniformly bounded if and only if $E \in \mathbf{CSP}$.

Let us prove that (3.25) holds if $\Omega_p^X(\mathbf{n})$ is uniformly bounded and $p \in acX$. In this case, as was proved above, $E \in \mathbf{CSP}$. Consequently, by Theorem 2.5, there is an universal element $\tilde{L} \in \tilde{I}_E^d$ such that $M(\tilde{L}) < \infty$. Lemma 3.9 implies that

$$M(\tilde{L}) = C_E. \quad (3.26)$$

By Proposition 3.10, we have also the equality

$$C_E = R^*(\Omega_0^E(\mathbf{n})). \quad (3.27)$$

Since $R^*(\Omega_0^E(\mathbf{n})) = R^*(\Omega_p^X(\mathbf{n}))$, equalities (3.26) and (3.27) imply (3.25). \square

Theorem 3.11 is an example of translation of some results related to completely strongly porous at 0 sets on the language of pretangent spaces. In the rest of the present section of the paper we shall obtain one more example.

Let $(X_i, d_i, p_i), i = 1, 2$, be pointed metric spaces. We write

$$(Y, d_y, p_y) = X_1 \uplus X_2$$

if (Y, d_y, p_y) is a pointed metric space for which there are $Y_i \subseteq Y$ and isometries $f_i : Y_i \rightarrow X_i$ such that $Y_1 \cup Y_2 = Y$, $p_y \in Y_1 \cap Y_2$ and $f_i(p_y) = p_i$ with $i = 1, 2$. For example, to construct $X_1 \uplus X_2$ we can use some isometric embeddings F_i of $X_i, i = 1, 2$, into a linear normed space X equipped with l_∞ norm [9, p. 13]. The translation $\Phi(x) = x + F_1(p_1) - F_2(p_2), x \in X$, is a self-isometry of X transforming $F_2(p_2)$ at $F_1(p_1)$. Consequently the

set $F_1(X_1) \cup \Phi(F_2(X_2))$ with the metric from X and the marked point $F_1(p_1) = \Phi(F_2(p_2))$ meets all desired properties of $X_1 \uplus X_2$.

Proposition 2.8 and Theorem 3.11 give us the following

Corollary 3.12. *Let (X, d_x, p_x) be a pointed metric space. The following statements are equivalent:*

- (i) $\Omega_{\mathbf{p}_z}^Z(\mathbf{n})$ is uniformly bounded for every $(Z, d_z, p_z) = Y \uplus X$ having the uniformly bounded $\Omega_{\mathbf{p}_y}^Y(\mathbf{n})$;
- (ii) $p_x \notin acX$.

For the proof it suffices to note that $S_{p_z}(Z) = S_{p_x}(X) \cup S_{p_y}(Y)$.

4 Uniform boundedness and uniform discreteness

Let $\mathfrak{F} = \{(X_i, d_i, p_i) : i \in I\}$ be a nonempty family of pointed metric spaces. We set

$$\rho_*(X_i) := \begin{cases} \inf\{d_i(x, p_i) : x \in X_i \setminus \{p_i\}\} & \text{if } X_i \neq \{p_i\} \\ +\infty & \text{if } X_i = \{p_i\} \end{cases} \quad (4.1)$$

for $i \in I$ and write $R_*(\mathfrak{F}) := \inf_{i \in I} \rho_*(X_i)$. We shall say that \mathfrak{F} is *uniformly discrete* (w.r.t. the marked points p_i) if $R_*(\mathfrak{F}) > 0$.

As in Proposition 3.1 it is easy to show that the family $\Omega_{\mathbf{p}}^X$ of all pre-tangent spaces is uniformly discrete if and only if p is an isolated point of the metric space X . Thus, it make sense to consider $\Omega_{\mathbf{p}}^X(\mathbf{n})$.

Theorem 4.1. *Let (X, d, p) be a pointed metric space such that $\Omega_{\mathbf{p}}^X(\mathbf{n}) \neq \emptyset$. Then $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly discrete if and only if it is uniformly bounded. The equality*

$$R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = \frac{1}{R^*(\Omega_{\mathbf{p}}^X(\mathbf{n}))} \quad (4.2)$$

holds if $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly bounded.

Remark 4.2. The condition $\Omega_{\mathbf{p}}^X(\mathbf{n}) \neq \emptyset$ implies that $R^*(\Omega_{\mathbf{p}}^X(\mathbf{n})) > 0$. Putting $\frac{1}{\infty} = 0$ we may also extend (4.2) on arbitrary nonvoid families $\Omega_{\mathbf{p}}^X(\mathbf{n})$.

We will prove Theorem 4.1 in some more general setting.

Let (X, d, p) be a pointed metric space and let $t > 0$. Write tX for the pointed metric space (X, td, p) , i.e. tX is the pointed metric space with the

same underlying set X and the marked point p , but equipped with the new metric td instead of d .

Definition 4.3. Let \mathfrak{F} be a nonempty family of pointed metric spaces. \mathfrak{F} is weakly self-similar if for every $(Y, d, p) \in \mathfrak{F}$ and every nonzero $t \in S_p(Y)$ the space $\frac{1}{t}Y$ belongs to \mathfrak{F} .

Theorem 4.4. Let $\mathfrak{F} = \{(X_i, d_i, p_i) : i \in I\}$ be a weakly self-similar family of pointed metric spaces. Suppose that the sphere

$$S_i = \{x \in X_i : d_i(x, p_i) = 1\}$$

is nonvoid for every $i \in I$. Then \mathfrak{F} is uniformly bounded if and only if \mathfrak{F} is uniformly discrete w.r.t. the marked points $p_i, i \in I$. If \mathfrak{F} is uniformly bounded, then the equality

$$R_*(\mathfrak{F}) = \frac{1}{R^*(\mathfrak{F})} \quad (4.3)$$

holds.

Proof. Assume that \mathfrak{F} is uniformly bounded but not uniformly discrete. Then there is a sequence $(x_{i_k})_{k \in \mathbb{N}}$ such that

$$x_{i_k} \in X_{i_k}, x_{i_k} \neq p_{i_k} \text{ and } \lim_{k \rightarrow \infty} d_{i_k}(x_{i_k}, p_{i_k}) = 0$$

Since $S_i \neq \emptyset$ for every $i \in I$, we can find a sequence $(y_{i_k})_{k \in \mathbb{N}}$ for which $y_{i_k} \in X_{i_k}$ and $d_{i_k}(y_{i_k}, p_{i_k}) = 1$ for every $k \in \mathbb{N}$. Define t_k to be $d_{i_k}(x_{i_k}, p_{i_k}), k \in \mathbb{N}$. Since \mathfrak{F} is weakly self-similar, the membership $t_k^{-1}X_{i_k} \in \mathfrak{F}$ holds for every $k \in \mathbb{N}$. Now we obtain

$$R^*(\mathfrak{F}) \geq \limsup_{k \rightarrow \infty} t_k^{-1} d_{i_k}(y_{i_k}, p_{i_k}) = \limsup_{k \rightarrow \infty} t_k^{-1} = \infty.$$

Hence \mathfrak{F} is not uniformly bounded, contrary to the assumption. Therefore if \mathfrak{F} is uniformly bounded, then \mathfrak{F} is uniformly discrete. Similarly we can prove that the uniform discreteness of \mathfrak{F} implies the uniform boundedness of this family.

Suppose now that $R^*(\mathfrak{F}) < \infty$. Let us prove equality (4.3). Define a quantity $Q(\mathfrak{F})$ by the rule

$$Q(\mathfrak{F}) = \sup_{i \in I} \frac{\rho^*(X_i)}{\rho_*(X_i)}$$

where $\rho_*(X_i)$ is defined by (4.1) and $\rho^*(X_i)$ by (3.1). The first part of the theorem implies that \mathfrak{F} is uniformly discrete. Hence $R_*(\mathfrak{F}) > 0$, that implies $\rho_*(X_i) > 0, i \in I$. Moreover the inequality $R^*(\mathfrak{F}) < \infty$ gives us the condition $\rho_*(X_i) < \infty$. Thus $Q(\mathfrak{F})$ is correctly defined. We claim that the equality $Q(\mathfrak{F}) = R^*(\mathfrak{F})$ holds. Indeed let $(i_k)_{k \in \mathbb{N}}$ be a sequence of indexes $i_k \in I$ such that

$$\lim_{k \rightarrow \infty} \rho_*(X_{i_k}) = R^*(\mathfrak{F}). \quad (4.4)$$

Since $S_i \neq \emptyset$ for every $i \in I$, we have $\rho_*(X_{i_k}) \leq 1$ for every X_{i_k} . Consequently

$$Q(\mathfrak{F}) \geq \limsup_{k \rightarrow \infty} \frac{\rho^*(X_{i_k})}{\rho_*(X_{i_k})} \geq \limsup_{k \rightarrow \infty} \rho^*(X_{i_k}) = \lim_{k \rightarrow \infty} \rho^*(X_{i_k}) \geq R^*(\mathfrak{F}). \quad (4.5)$$

Let us consider a sequence $(i_m)_{m \in \mathbb{N}}, i_m \in I$, for which

$$Q(\mathfrak{F}) = \lim_{m \rightarrow \infty} \frac{\rho^*(X_{i_m})}{\rho_*(X_{i_m})}. \quad (4.6)$$

The quantity $\frac{\rho^*(X_i)}{\rho_*(X_i)}$ is invariant w.r.t. the passage from X_i to $\frac{1}{t}X_i, t \in S_{p_i}(X_i)$. Consequently using the uniform discreteness of \mathfrak{F} and the inequality $\rho_*(X_i) \leq 1$ (which follows from the condition $S_i \neq \emptyset, i \in I$), we may assume that

$$\lim_{m \rightarrow \infty} \rho_*(X_{i_m}) = 1. \quad (4.7)$$

Limit relations (4.6) and (4.7) imply

$$Q(\mathfrak{F}) = \lim_{m \rightarrow \infty} \rho_*(X_{i_m}) \leq R^*(\mathfrak{F}).$$

The last inequality and (4.5) give us the equality $R^*(\mathfrak{F}) = Q(\mathfrak{F})$. Reasoning similarly we obtain the equality $Q(\mathfrak{F}) = \frac{1}{R_*(\mathfrak{F})}$. Equality (4.3) follows. \square

Let us define a subset ${}^1\Omega_{\mathbf{p}}^{\mathbf{X}}$ of the set $\Omega_{\mathbf{p}}^{\mathbf{X}}$ of all pretangent spaces to X at p by the rule:

$$\begin{aligned} (\Omega_{p, \tilde{r}}^X, \rho, \alpha) &\in {}^1\Omega_{\mathbf{p}}^{\mathbf{X}} \quad \text{if and only if } \tilde{r} \text{ is almost decreasing} \\ \text{and } \quad \{\delta \in \Omega_{p, \tilde{r}}^X : \rho(\alpha, \delta) = 1\} &\neq \emptyset \end{aligned}$$

where $\alpha = \pi(\tilde{p})$ is the marked point of the pretangent space $\Omega_{p,\tilde{r}}^X$ and ρ is the metric on $\Omega_{p,\tilde{r}}^X$. It is clear that ${}^1\Omega_{\mathbf{p}}^X$ meets the condition of Theorem 4.4 and

$${}^1\Omega_{\mathbf{p}}^X \subseteq \Omega_{\mathbf{p}}^X(\mathbf{n}). \quad (4.8)$$

To apply Theorem 4.4 to Theorem 4.1 we need the following

Lemma 4.5. *Let $\Omega_{p,\tilde{r}}^X \in \Omega_{\mathbf{p}}^X(\mathbf{n})$. Then there is ${}^1\Omega_{p,\tilde{\mu}}^X \in {}^1\Omega_{\mathbf{p}}^X$ such that*

$$\rho_*({}^1\Omega_{p,\tilde{\mu}}^X) \leq \rho_*(\Omega_{p,\tilde{r}}^X) \leq \rho^*(\Omega_{p,\tilde{r}}^X) \leq \rho^*({}^1\Omega_{p,\tilde{\mu}}^X). \quad (4.9)$$

Proof. Let $\tilde{X}_{p,\tilde{r}}$ be the maximal self-stable family which metric identification coincides with $\Omega_{p,\tilde{r}}^X$. We can find some sequences $\tilde{a}^i = (a_n^i)_{n \in \mathbb{N}}$ and $\tilde{b}^i = (b_n^i)_{n \in \mathbb{N}}$, $i \in \mathbb{N}$ such that

$$\lim_{i \rightarrow \infty} \rho(\pi(\tilde{b}^i), \alpha) = \rho_*(\Omega_{p,\tilde{r}}^X) \quad \text{and} \quad \lim_{i \rightarrow \infty} \rho(\pi(\tilde{a}^i), \alpha) = \rho^*(\Omega_{p,\tilde{r}}^X). \quad (4.10)$$

Since $\Omega_{p,\tilde{r}}^X \in \Omega_{\mathbf{p}}^X(\mathbf{n})$, the scaling sequence \tilde{r} is normal. Consequently there is $\tilde{c} = (c_n)_{n \in \mathbb{N}} \in \tilde{X}$ such that

$$\lim_{n \rightarrow \infty} \frac{d(c_n, p)}{r_n} = 1.$$

Let us define a countable family $\mathfrak{B} \subseteq \tilde{X}$ as

$$\mathfrak{B} = \{\tilde{b}^i : i \in \mathbb{N}\} \cup \{\tilde{a}^i : i \in \mathbb{N}\} \cup \{\tilde{c}\}.$$

The family \mathfrak{B} satisfies the condition of Lemma 1.5. Consequently there is an infinite subsequence $\tilde{r}' = (r_{n_k})_{k \in \mathbb{N}}$ the scaling sequence \tilde{r} for which the family

$$\mathfrak{B}' = \{(b_{n_k}^i)_{k \in \mathbb{N}} : i \in \mathbb{N}\} \cup \{(a_{n_k}^i)_{k \in \mathbb{N}} : i \in \mathbb{N}\} \cup \{(c_{n_k})_{k \in \mathbb{N}}\}$$

is self-stable w.r.t. \tilde{r}' . Completing \mathfrak{B}' to a maximal self-stable family and passing to the metric identification of it we obtain the desired pretangent space ${}^1\Omega_{p,\tilde{\mu}}^X$ with $\tilde{\mu} = \tilde{r}'$. \square

Proof of Theorem 4.1. Inclusion (4.8) implies the inequalities

$$R^*({}^1\Omega_{\mathbf{p}}^X) \leq R^*(\Omega_{\mathbf{p}}^X(\mathbf{n})) \quad \text{and} \quad R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})) \leq R_*({}^1\Omega_{\mathbf{p}}^X).$$

The converse inequalities follow from (4.9). Consequently we have the equalities

$$R^*({}^1\Omega_{\mathbf{p}}^X) = R^*(\Omega_{\mathbf{p}}^X(\mathbf{n})) \quad \text{and} \quad R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = R_*({}^1\Omega_{\mathbf{p}}^X). \quad (4.11)$$

Now Theorem 4.1 follows directly from Theorem 4.4 and (4.11). \square

The sum of theorems 4.1 and 3.11 yields the following result that was announced in [3].

Theorem 4.6. *Let (X, d, p) be a metric space with a marked point $p \in acX$. Then the following three conditions are equivalent.*

- (i) $\Omega_p^X(\mathbf{n})$ is uniformly bounded.
 - (ii) $\Omega_p^X(\mathbf{n})$ is uniformly discrete.
 - (iii) $S_p(X) \in \mathbf{CSP}$.
- Moreover if $\Omega_p^X(\mathbf{n})$ is uniformly bounded, then

$$R^*(\Omega_p^X(\mathbf{n})) = M(\tilde{L}) \quad \text{and} \quad R_*(\Omega_p^X(\mathbf{n})) = \frac{1}{M(\tilde{L})}$$

where the quantity $M(\tilde{L})$ was defined by (2.2).

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